UNIT 1 : REAL NUMBER SYSTEM- I

UNIT STRUCTURE

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1.1 LEARNING OBJECTIVES

After going through this unit, you will able to know -

- field axioms, properties of real numbers, absolute value (or modulus of a real number
- completeness property of real numbers
- archimedean property of real numbers
- density property of real numbers.

1.2 INTRODUCTION

Real analysis is an indispensable branch of Mathematics. It deals with the study of real numbers. Generally calculus emphasizes learning the methods of differentiation and integration. In real analysis we study the principles that provide these methods.

In this unit, we will study field axioms, properties of real numbers, absolute value (or modulus) of a real number; completeness property of real numbers; Archimedean property of real numbers; density property of real numbers.

1.3 FIELD AXIOMS

Let S be a nonempty set. A rule, which associates every pair of elements of S with a unique element of S, is called a binary operation on S.

We know that the sum of two natural numbers is always a natural number. So, addition is a binary operation on the set N of natural numbers. Similarly multiplication is a binary operation on N, but subtraction and division are not binary operations on N.

A nonempty set F equipped with two binary operators defined on it, denoted by the symbols '+' (addition) and " . " (multiplication), is called a field if the following axioms are satisfied.

A1 : $a + b \in F, \forall\ a, b \in F$

i.e; F is closed under "+"

A2 : $(a + b) + c = a + (b + c) \ \forall\ a, b, c, \in F$,

i.e; "+" is associative

A3 : There exists an element 0 in F, called the zero or the zero element of F such that $a + 0 = a = 0 + a, \ \forall\ a \in F$

A4 : For every $a \in F$, there exists an element $-a \in F$ such that

$a + (-a) = 0 = (-a) + a$

–a is called addivitive inverse of a

A5 : $a + b = b + a, \ \forall\ a, b \in F$

i.e; "+" is commutative

M1 : $a. b \in F, \ \forall\ a, b \in F$

i.e., F is closed undder "."

M2 : $(a . b). c = a. (b . c)$

i.e., " . " is associative.

M3 : There exists an element 1 in F different from 0, called the unity of F suc that

a.1 = a = 1.a, $\forall$ a $\in$ F

M4 : For every a $\in$ F, a $\neq$ 0, there exists an element $a^{-1} \in$ F such that

$a.a^{-1} = a = a^{-1}.a$

a^{-1} is called multiplicative inverse of a

M5 : a . b = b. a, $\forall$ a, b $\in$ F

i.e., " . " is commutative

D : a. (a + c) = a. b + a. c, (left distributive law0

(a + b). c = a, c + b. c, (right distributive law)

$\forall$ a, b, c $\in$F

Note

1. A nonempty set, equipped with one or more binary operations defined on it, is called an algebraic structure. Clealy a field is an algebraic structure.
2. In a field unity is not equal zero, i.e., 1 $\neq$ 0. So, a field must have at least two elements.
3. It is clear from the definition of binary operation that F is closed under '' + '' and " . ". So, the axioms A1 and M1 are superfluous.
4. In a field, additive inverse as well as multiplicative inverse of an element is unique.
5. In a field, the zero and the unity are unique.

Example of Field

1. The set $\underline{\underline{Q}}$ of rational numbers is a field under ordinary addition and multiplication as the eleven axioms of a field are satisfied.
2. The set of $\underline{\underline{R}}$ of real numbers is a field under ordinary addition and multiplication.

Note :

The set $\underline{\underline{N}}$ of natural numbers is not a field. We have 5 $\in \underline{\underline{N}}$ but – 5 $\notin \underline{\underline{N}}$

The set $\underline{\underline{Z}}$ of integers is not a field. We have $5 \in \underline{\underline{Z}}$, but $\frac{1}{5} \notin \underline{\underline{Z}}$.

Remark : In the subsequent discussions, we will omit the symbol ".". We will write ab instead of a.b.

1.4 PROPERTIES OF REAL NUMBERS

1.4.1 Algebraic properties of real numbers

Theorem 1.1 Let a, b, c be real numbers. Then

(i) a + b = a + c implies b = c

(Left cancellation law for addition)

(ii) ab = ac implies b = c, $a \neq 0$

(Left cancellation law for multiplication)

Proof :

(i) We have $a \in \underline{\underline{R}}$

$\therefore -a \in \underline{\underline{R}}$

a + b = a + c

$\Rightarrow$ (–a) + (a + b) = (–a) + (a + c)

$\Rightarrow$ (– a + a) + b = (– a + a) + c, by A2

$\Rightarrow$ 0 + b = 0 + c, by A4

$\Rightarrow$ b = c

(ii) We have $a \in \underline{\underline{R}}$, $a \neq 0$

$\therefore a^1 \in \underline{\underline{R}}$

ab = ac

$\Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$, by M2

$\Rightarrow$ 1b = 1c, by M4

$\Rightarrow$ b = c

Remark : Similarly we can prove that

(iii) b + a = c + a implies b = c

(Right cancellation law for addition)

(iv) ba = ca implies b = c, $a \neq 0$

(Right cancellation law for multiplication)

Theorem 1. 2 Let a be any real

Then (i) a0 = 0 = 0a

(ii) (–1) a = – a = a (–1)

(iii) – (–a) = a

(iv) $(a^{-1}) = a, a \neq 0$

Proof : (i) We have $0 + 0 = 0$

$\Rightarrow a (0 + 0) = a0$

$\Rightarrow a0 + a0 = a0$, by D

$\Rightarrow a0 + a0 = a0 + 0$

$\Rightarrow a\ 0 = 0$

(by left cancellation law for addition

similarly $0a = 0$

(ii) We have $1+(-1)=0$

$\Rightarrow (1 + (-1)\ a = 0\ a$

$\Rightarrow (1\ (-1))\ a = 0$

$\Rightarrow 1a + (-1)\ a = 0$

$\Rightarrow a + (-1)\ a = 0$

$\Rightarrow (-a) + (a + (-1)\ a) = (-a) + 0, [\ a \in \underline{\underline{R}} \Rightarrow -a \in \underline{\underline{R}}\ ]$

$\Rightarrow ((-a) + a) + (-1)\ a = -a$

$\Rightarrow \quad 0 + (-1)\ a = -a$

$\Rightarrow \quad (-1)a = -\ a$

similarly, We can prove that

$a\ (-1) = -\ a$

(iii) We have $a + (-a) = 0$, by A4 ... (1)

Replacing a by $-\ a$, we get

$-\ a + (-(-a)) = 0$ ... (2)

From (1) and (2)

$-\ a + (-(-a)) = a + (-a)$

$\Rightarrow -\ a + (-(-a)) = (-a) + a$, by A5

$\Rightarrow (-\ (-a)) = a,$

by lefy cancellation law for addition.

(iv) We have $aa^{-1} = 1, a \neq 0$ by M4 ... (1)

Replacing a by a^{-1}, We get

$a^{-1}\ (a^{-1})^{-1} = 1$ ... (2)

From (1) and (2)

$a^{-1}\ (a^{-1})^{-1} =\ a\ a^{-1}$, by M5

$\Rightarrow a^{-1}(a^{-1})^{-1} = a^{-1}a$

$\Rightarrow (a^{-1})^{-1} = a$

by left cancellation law for multiplication.

Theorem 1.3 Let a, b be real numbers then

(i) $a(-b) = -(ab) = (-a)b$

(ii) $(-a)(-b) = ab$

Proof (i) We have $b + (-b) = 0$, by A4

$\therefore\ a(b + (-b) = a0$

$\Rightarrow ab + a(-b) = 0$, ny D and Theorem 1.2 ... (1)

Again $-(ab)$ is a real number

$\therefore\ -(ab) + (ab + a(-b)) = -(ab) + 0$

$\Rightarrow (-(ab) + (ab)) + a(-b) = -(ab)$, by A2, A3

$\Rightarrow 0 + a(-b) = -(ab)$, by A4

$\Rightarrow a(-b) = -(ab)$, by A3

Similarly $(-a)b = -(ab)$

(ii) We have $a(-b) = -(ab)$

Replacing a by (–a), We get

$(-a)(-b) = -((-a)b)$

$\Rightarrow (-a)(-b) = -(-(ab))$, Theorem 1.3 (i)

$\Rightarrow (-a)(-b) = ab$, Theorem1.2(iii)

1.4.2 Order properties of real numbers

We know that every real number can be represented by a unique point on line, and, conversely, every point on a line represents a unique real number.

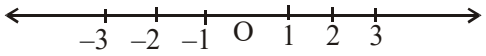


Fig 1.1

A real number a $\neq$ 0 said to be positive if it lies on the right side of the point 0 which corresponds to the real number zero. We write it as $a > 0$. Similarly a real number a is said to be negative if it lies on the left side of 0. We write it as $a < 0$

If $a \geq 0$ thew a is said to be nonnegative. Similarly a is said to be nonpositive if $a \leq 0$.

Let a and b be any two real numbers.

We say that a is greater than b if a – b > 0. We write it as a > b.

Similarly a is less than b if a – b < 0. We write it as a < b.

a is greater than or equal to b if a – b $\geq$ 0. We write it as a $\geq$ b.

Similarly a is less than or equal to b if a – b $\leq$ 0. We write it as a $\leq$ b.

The order relation < (0r >) and $\underline{\underline{R}}$ satisfies the following conditions.

01. Trichotomy Law

Let a, b be two real numbers. Then exactly one of the following holds.

(i) a is greater than (a > b).

(ii) a is equal to be (a = b)

(iii) a is less than b (a < b)

0.2 Order is transitive

Let a, b, c be real numbers such that a < b, and b < c. Then acc

03. Order is compatible with addition

a < b $\Rightarrow$ a + c < b + c, for any real numbers a, b, c.

04. ordering is compatible with multiplication

(i) a < b and c > 0 $\Rightarrow$ ac < bc

(ii) a < b and c < 0 $\Rightarrow$ ac < bc.

The field of real numbers $\underline{\underline{R}}$ together with the order relation defined on $\underline{\underline{R}}$ satisfying 01, 02, 03, 04 becomes an ordered field.

Note 1. The field of rational numbers $\underline{\underline{Q}}$ is ordered field.

2. The set $\underline{\underline{C}}$ of complex numbers is a field, but it is not ordered.

Order cannot be defined on $\underline{\underline{C}}$.

i = $\sqrt{-1}$ is neither positive nor negative

i $\neq$ 0, i is not a real number.

Do You Know?

The meaning of the word trichotomy is 'division into three categories'. In theology, it means 'the division of human beings into body, spirit, and soul.'

Source : Collins English Dictionary

Theorem 1.4 Let a, b be real numbers. Then

(i) $a > 0 \Rightarrow -a < 0$

(ii) $a < 0 \Rightarrow -a > 0$

(iii) $a > 0, b > 0 \Rightarrow a + b > 0$

(iv) $a < 0, b < 0 \Rightarrow a + b < 0$

(v) $a > 0, b > 0 \Rightarrow ab > 0$

(vi) $a < 0, b < 0 \Rightarrow ab > 0$

(vii) $a > 0, b < 0 \Rightarrow ab < 0$

Proof

(i) We have $a \in \underline{\underline{R}}$ and a>0

Now $a > 0, a + (-a) = 0$

By the law of trichotomy, $-a < 0$ or $-a=0$, or $-a > 0$

If possible, let $-a > 0$

$\therefore (-a) + a > 0 + a$, by 03

or $0 > a$,

Which is a contradication.

Again a = 0 is a contradiction.

$\therefore -a < 0$

(ii) Proceed as above

(iii) We have

$a > 0$

$\therefore a + b > 0 + b$, by 03

$\Rightarrow a + b > b$

Now $a + b > b$, and $b > 0$

$\therefore$ by 02, $a + b > 0$

(iv) Proceed as above

(v) We have $a > 0, b > 0$

$\therefore ab > ob$ by 04 (i)

$\Rightarrow ab > 0$

(vi) We have $a < 0, b < 0$

$\therefore a < 0, -b > 0$

$\Rightarrow$ $a(-b) < 0(-b)$, by 04 (i)

$\Rightarrow$ $a(-b) < 0(-b)$, by 04 (i)

$\Rightarrow$ – ab < 0

$\Rightarrow$ – (–ab) > 0, by (iii)

$\Rightarrow$ ab < 0

(vii) Try yourself.

Intervals

Let a, b be real numbers.

(i) The set {x : a $\leq$ x $\leq$ b} denoted by [a, b], is called a closed interval. The end points a, b are included.

(ii) The set {x : a < x < b}, denoted by]a, b[, is called an open interval. The end point a, b are not included.

(iii) The set {x : a < x $\leq$ b} is denoted by]a, b].

The set {x : a $\leq$ x < b } is denoted by [a, b[.

These two are called semi-clossed or semi-open intervals.

The former is open at a, and closed at b; while the latter is closed at a and open at b.

1.5 ABSOLUTE VALUE (OR MODULUS) OF A REAL NUMBER

Let x be any real number.

The absolute value (or the modulus) of x denoted by |x|, is defned by

$$|x| = \begin{cases} x, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -x, \text{if } x < 0 \end{cases}$$

For example |10| = 10, |0| = 0, |–10| = – (– 10) = 10

$|x| \geq 0$, for all $x \in \underline{\underline{R}}$

Note :

1. |x| is a nonnegative real number.

2. |–x| $\neq$ – |x|

3. |x| = 0 if and only if x = 0

Properties

1. If x is any real number,

|x| = max {x, – x}, where max {x, –x} denotes the greater of the two numbers x, – x.

Proof : x is any real number.

Therefore, there are three possibilities :

$x > 0$, or $x = 0$, or $x < 0$

case 1. Let $x > 0$

$\therefore \quad -x < 0$

$x > -x$

since $x > 0$, $|x| = x$

Maximum of $\{x , -x\} = x$

$\therefore |x| = \max (x, -x\}$

case 2 Let $x = 0$

$\therefore \quad x = 0 = -x$

$|x| = 0$

maximum of $\{x , -x\} = 0$

$\therefore |x| = \max \{x, -x\}$

case 3. Let $x < 0$

$\therefore \quad -x > 0$

$\therefore \quad -x > x$

Since $x < 0$, $|x| = -x$

Maxmimum of $\{x, -x\} = -x$

$\therefore |x| = \max \{x, -x\}$

Thus, for all $x \in \underline{\underline{R}}$,

$|x| = \max \{x, -x\}$

Note : It follows that $x \leq |x|, -x \leq ||x||$

2. If x is any real number, $|x| = |-x|$

Proof : We have $|x| = \max \{x, -x)$

$\therefore |x| = \max \{ -x, -(-x)\}$

$= \max \{-x, x\}$

$= \max \{x, -x\}$

$= |x|.$

3. If x is any real number, then $|x|^2 = x^2 = |-x|^2$

Proof: case 1. Let $x > 0$

$\therefore |x| = x$

$\therefore |x|^2 = |x|\,|x| = xx = x^2$

case 2. Let x = 0

$\therefore |x| = 0$

clearly $|x|^2 = x^2$

case 3. Let $x < 0$

$\therefore |x| = -x$

$\therefore |x|^2 = |x|\,|x| = (-x)(-x) = x^2$

Thus for all $x \in \underline{\underline{R}}$,

$|x|^2 = x^2 = |-x|^2$

4. If x, and y are any two real numbers, then

$|x < t| \leq |x| + |y|$

[The above inequality is known as the triangle inequality.]

Proof : x and y are any two real numbers.

$\therefore$ x + y is also a real number.

Therefore, there are three possibities, x + y > 0, or, x + y = 0, or, x + y < 0.

Let $x + y > 0$

$\therefore$ by definition $|x + y| = x + y$ ...(1)

Again, we know that

$x \leq |x|$, for all $x \in \underline{\underline{R}}$,

$y \leq |y|$, for all $y \in \underline{\underline{R}}$,

$\therefore$ $x + y \leq |x| + |y|$ ...(2)

For (1) and (2), if follows that

$|x + y| \leq |x| + |y|$

Let, x + y = 0

Then, by definition |x + y | = 0

$\therefore |x + y| = x + y$

Let $x + y < 0$

$\therefore$ by definition $|x + y| = -(x + y)$

$= (-x) + (-y)$ ...(3)

Again we know that

$-x \leq |x|$, for al $x \in \underline{\underline{R}}$

$-y \leq |y|$, for all $y \in \underline{\underline{R}}$

$\therefore$ $(-x) + (-y) \leq |x| + |y|$ ... (4)

From (3) and (4), it folowing that

$|x + y| \leq |x| + |y|$.

Thus for all x, $y \in \underline{\underline{R}}$, $|x + y| \leq |x| + |y|$

5. If x and y are any two real numbers,

Then $|x - y| \geq ||x| - |y||$

Proof We have

$x = (x - y) + y$

$\therefore |x| = |(x - y) + y|$

$\leq |x - y| + |y|$, by triangle inequality

$\therefore \quad |x| - |y| \leq |x - y|$ ... (1)

Interchanging x and y in (1), we get

$|y| - |x| \leq |y - x|$

Now $|y - x| = |-(x-y)| = |x - y|$

$\therefore \quad |y| - |x| \leq |x - y|$

or $\quad -(|x| - |y|) \leq |x - y|$ ... (2)

From (1) and (2), it folows that $|x| - |y|$ and its negative $-(|x| - |y|)$ are both less than or at most equal to $|x - y|$

$\therefore \max \{|x| - |y|, -(|x| - |y|)\} \leq |x - y|$

This implies $||x| - |y|| \leq |x - y|$,

i.e., $|x - y| \geq ||x| - |y||$

Thus, for all x. $y \in \underline{\underline{R}}$,

$|x\ y| \geq ||x| - |y||$

6. It x and y are any two real numbers, then

$|xy| = |x|\,|y|$

Proof : x and y are any two real numbers.

There are three possibities,

$x \geq 0, y \geq 0$

Or

x and y have opposite signs

Or

$x \leq 0, y \leq 0$

Let $x \geq 0, y \geq 0$

$\therefore \quad |x| = x, |y| = y$

$\therefore$ $x \geq 0, y \geq 0 \therefore xy \geq 0$

$\therefore$ $|xy| = xy = |x|\,|y|$

Let x and y have opposite signs.

Let $x > 0\ y < 0$

$\therefore$ $|x| = \max\{x, -x\} = x$

$\therefore$ $|y| = \max\{y, -x\} = -y$

$\because$ $x > 0, y < 0 \therefore xy < 0$

$\therefore$ $|xy| = -(xy) = x(-y) = |x|\,|y|$

Let $x \leq 0, y \leq 0$

$\therefore$ $|x| = -x, |y| = -y$

$\therefore$ $x \geq 0, y \leq 0 \therefore xy \geq 0$

$\therefore$ $|xy| = xy = (-x)(-y) = |x|\,|y|$

Thus, for all $x, y \in \underline{\underline{R}}$,

$|xy| = |x|\,|y|$

7. If x is any non zero real number, then $\left|\frac{1}{x}\right| = \frac{1}{|x|}$

Proof : Let $x > 0$

$\therefore |x| = x$

Also, $\frac{1}{x} > 0$

$\therefore$ $\left|\frac{1}{x}\right| = \frac{1}{x} = \frac{1}{|x|}$

Let $x < 0$

Also $\frac{1}{x} < 0$

$\therefore$ $\left|\frac{1}{x}\right| = -\frac{1}{x} = \frac{1}{|-x|} = \frac{1}{|x|}$

Thus, for any nonzero real number x,

$$\left|\frac{1}{x}\right| = \frac{1}{|x|}$$

8. If x is a real number and y is a nonzero real number, then

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

Proof :

$$\left|\frac{x}{y}\right| = \left|x.\frac{1}{y}\right|$$

$$= |x| \left|\frac{1}{y}\right| \text{ (Property 6)}$$

$$= |x| \left|\frac{1}{|y|}\right| \text{ (Property 7)}$$

$$= \frac{|x|}{|y|}$$

9. If x and y are any two real numbers, then

$|x + y|^2 + |x – y|^2 = 2 (|x|^2 + |y|^2)$

Proof: We have

$|x + y|^2 = (x + y)^2$, (Property 3)

$|x – y|^2 = (x – y)^2$, (roperty 3)

$\therefore \quad |x + y|^2 + |x – y|^2 = (x + y)^2 + (x – y)^2$

$= 2 (x^2 + y^2)$

$= 2 (|x|^2 + |y|^2)$

10. If x and y are any two real numbers and $\delta > 0$ then

$|x – a| < \delta$ if and only $a – \delta < x < a + \delta$

Proof: We have

$|x – a| = \max \{(x – a), – (x – a)\}$

$= \max \{(x – a), a – x)\}$

Now, $|x – a| < \delta$

$\Leftrightarrow \max \{(x – a), (a – x)\} < \delta$

$\Leftrightarrow x – a < \delta$ and $a – x < \delta$

$\Leftrightarrow x < a + \delta$ and $a – \delta < x$

$\Leftrightarrow a – \delta < x < a + \delta$

11. If x is any real number, and $\delta < 0$, then $|x| < \delta$ if and only if $– < \delta < x < \delta$

Proof: We have $|x| = \max \{x , – x\}$

$|x| < \delta$

$\Leftrightarrow \max \{(x – x) < \delta$

$\Leftrightarrow x < \delta$ and $– x < \delta$

$\Leftrightarrow x < \delta$ and $– \delta < x$

$\Leftrightarrow -\delta < x < \delta$

Example 1. Let x and y be two real numbers. Show that

$|x + y| = |x| + |y|$

if and only if $xy \geq 0$

Solution : $|x + y| = |x| = |y|$

$\Leftrightarrow |x + y|^2 = (|x| + |y|)^2$

$\Leftrightarrow |x + y|^2 = (|x| + |y|)^2$

$\Leftrightarrow (x + y)^2 = |x|^2 + 2|x|\,|y| + |y|^2$, $[|x|^2 = x^2]$

$\Leftrightarrow x^2 + 2xy + y^2 = x^2 + 2|xy| + y^2$

$\Leftrightarrow xy = |xy|$

$\Leftrightarrow xy \geq 0$, by definition.

Example 2. Write the inequality $6 < x < 8$ in the absolute value form.

Solution : We know that $|x - a| < \delta$ if and only if $x - \delta < x < a + \delta$.

$6 < x < 8$ (given)

$\Rightarrow 7 - 1 < x < 7 + 1$

$\Rightarrow |x - 7| < 1$

Example 3. Rewrite the inequality $|x - 4| < 1$ by removing the absolute value symbol.

Solurion : We know that $|x - a| < \delta$ if and only if $a - \delta < x < a + \delta$

$|x - 4| > 1$

$\Rightarrow 4 - 1 < x < 4 + 1$

$\Rightarrow 3 < x < 5$

Example 4. If in the inequality $|x - a| < \delta$, $a = 5$, $\delta = 0.01$, write down any three values of x satisfying $|x - a| < \delta$

Solution : We know that $|x - a| < \delta$ if and only if $a - \delta < x < a + \delta$

Here $a = 5$, $\delta = 0.01$

$\therefore |x - 5| < .01$

$\Rightarrow 5 - 0.01 < x < 5 + 0.01$

$\Rightarrow 4.99 < x < 5.01$

$\therefore$ any three values of x satisfting the above inequality are 4.999, 5, 5.001.

Example 5. Solve $\left|\dfrac{x+2}{x-6}\right| = 5$

Solution : $\left|\frac{x+2}{x-6}\right| = 5$ given

$\therefore \left|\frac{x+2}{x-6}\right| = 5$ or $\left|\frac{x+2}{x-6}\right| = -5$

Now $\left|\frac{x+2}{x-6}\right| = 5 \quad \Rightarrow x + 2 = 5 (x - 6)$

$\Rightarrow x + 2 = 5x - 30$

$\Rightarrow 4x = 32$

$\therefore x = 8$

Again $\left|\frac{x+2}{x-6}\right| = -5 \quad \Rightarrow x + 2 = -5 (x - 6)$

$\Rightarrow x + 2 = -5x + 30$

$\Rightarrow 6x = 28$

$\therefore x = \frac{28}{2} = \frac{14}{3}$

$\therefore \quad x = 8, \frac{14}{3}$

Absoluts Value Function

Using the concept of absolute value of a real number, the absolute value function (or the modulus function) can be defined as follows

$$f(x) = |x| = \begin{cases} x, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -x, \text{if } x < 0 \end{cases}$$

The graph of the function is shown in Fig 1.2

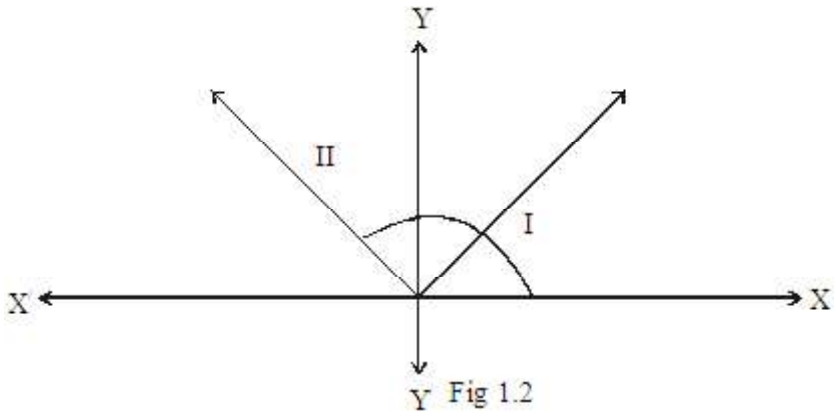


Fig 1.2

Note : 1. The graph consists of two rays, both starting at the origin.

2. The ray I marks an angle of $\frac{\pi}{4}$ with the positive direction of the x–

axis. The ray II markes an angle of $\frac{3\pi}{4}$ with the positive direction of the x - axis. The two rays are perpendicular to each other.



CHECK YOUR PROGRESS

Q.1. If x, y, z are any three real numbers, prove that

(i) $|x + y + z| \leq |x| + |y| + |x|$

(ii) $|xyz| = |x|\,|y|\,|z|$

Note : The above results can be extended to any finite number of real numbers. Thus if $x_1, x_2, x_3 \ldots, x_4$ are any real numbers then

(i) $|x_1 + x_2 + x_3 + \ldots\ldots + x_4| \leq |x_1| + |x_2| + |x_3| + \ldots\ldots + |x_4|$

(i) $|x_1\ x_2\ x_3 \ldots\ldots x_4| = |x_1|\ |x_2|\ |x_3|\ \ldots\ldots |x_4|]$

Q. 2. If in the inequality $|x - a| < \delta$, a = 9, δ =0.002, write down any three values of x satisfying $|x - a| < \delta$.

Q. 3. Write the inequality $10 < x < 12$ in the absolute value form.

Q. 4. Rewrite the inequality $|x - 7| < 0.0001$ by removing the absolute value symbol.

Q. 5. If x, y are any two real numbers, prove that $|x - y| \leq |x| + |y|$

Q.6. Show that $a < x < b$ if and only if $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$

Q. 7. Show that for any two real numbers a and b.

(i) $\max\{a, b\} = \frac{1}{2}[a + b + |a - b|]$

(ii) $\min\{a , b\} = \frac{1}{2}[a + b - |a - b|]$

Q. 8. Solve $\left|\frac{x+5}{3x-1}\right| = 4$

Q. 9. Show that $|a + b| = |a| + b|$ if and only if $ab \geq 0$.

Q.10. Show that $|a + b| < |a| + |b|$ if and only $ab < 0$

Q. 11. Which of the following statements are true ?

(i) $\{x : |x - 5| < 2\} = \{x : 3 < x < 7\}$

(ii) $\{x : |x - 4| < 1\} = \{x : -1 < x < 1\}$

(iii) $|x : |x - 1| < 2\} = \{x : 1 < x < 3\}$.

Q.12. If $f(x) = \frac{|x|}{x}$, and a is a nonzero real number, show that $|f(a) - f(-a)| = 2$

Q.13. If a and b are real numbers, show that

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

Q.14. If a and b are real numbers, show that

$$\frac{|a-b|}{1+|a-b|} = \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

1.6 COMPLETENESS PROPERTY OF THE REAL NUMBERS

Let us consider the set

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots\right\}$$

We see that each element of the set S is greater than zero. Then we say that 0 is a lower bound of S. Similary –1, – 2, etc. are also lower bounds of S.

Again, we see that each element of S in less than or equal to one. We say that 1 is an upper bound of S. Similarly 2, 3, etc. are upper bounds of S

Again, we see that 0 is the greatest of all lower bounds and 1 is the least of al upper bounds.

Definitions : Let S be a subset of $\underline{\underline{R}}$.

(i) S is bounded above if there exists $u \in \underline{\underline{R}}$ such that $x \leq u$ for all $x \in S$. u is an upper bound of S.

(ii) S if bounded below if there exists $w \in \underline{\underline{R}}$ such that $w \leq x$ for all $x \in S$. w is a lower bound of S.

(iii) S is bounded if it is both bounded above and bounded below.

(iv) S is unbounded if it is not bounded.

(v) Let S be bounded above. An upper bound u of S is said to be least upper bound (or supremum) of S if $u \leq u'$ for any upper bound of S. Symbolically we write u =sup(S) = lub (S).

(vi) Let S be bounded below. A lower bound w of S is said to be greatest lower bound (or infimum) of S if $w' \leq w$ for any lower bound w' of S.

Symblically we write w = inf (S) = glb (S)

Note : Any finite set is bounded

Examples (i) Let us consider the set

$$S = \left\{ \frac{n}{n+1} : n \in \underline{\underline{N}} \right\}$$

$$= \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ... \right\}$$

Clearly S is bounded

lub (S) = sup(S) = 1

glb (S) = inf (S) = $\frac{1}{2}$

(ii) Let us consider the intervals [a, b],]a, b[,]a, b[

Each of these intervals is bounded

lub = a, glb = b

(iii) The set { $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, ...} is bounded.

(iv) The set $\underline{\underline{N}}$ of natural numbers is bounded below, but not bounded above.

(v) The set $\underline{\underline{Z}}$ of integers is unbounded.

(vi) The set $\underline{\underline{Q}}$ of rational numbers is unbounded.

(vii) The set $\underline{\underline{R}}$ of real numbers is unbounded.

An empty set is bounded. Every real number x is an upper bound. Every real number x is a lower bound.

Example 6 : Let $A \subseteq \underline{\underline{R}}$, B $\subseteq \underline{\underline{R}}$ such that $A \subseteq B$.

(i) prove that if B is bounded above (below), then A in bounded above (below).

(ii) Give an example of sets A and B such that A is bounded below, but B is not bounded below.

Solution (i) Let B be bounded above

$\therefore$ there exists $u \in \underline{\underline{R}}$ such that $x \leq u$, for all $x \in B$

$\therefore$ $y \leq u$, for all $y \in A$.

$\therefore$ A is bounded above.

Similarly, it can be shown that if B is bounded below, then A is bounded below.

(ii) Let A = $\underline{\underline{N}}$, the set of natural numbers, and $B = \underline{\underline{Z}}$, the set of integers clearly, $A \subseteq B$.

Moreover A is bounded below, but B is not bounded.

Theorem 1.5: (i) Union of two bounded sets is bounded.

(ii) Intersection of two bounded sets is bounded.

Proof : (i) Let $A \subseteq \underline{\underline{R}}$, $B \subseteq \underline{\underline{R}}$ such that A. B are bounded.

A is bounded $\Rightarrow$ A is bounded above, and bounded below.

B is bounded $\Rightarrow$ B is bounded above, and bounded below.

$\therefore$ There exist w_1 (lower bound) and u_1 (upper bound) such that

$w_1 \leq x \leq u_1$, for all $x \in A$.

Again, there exist w_2 (lower bound) and u_2 (upper bound) such that

$w_2 \leq x \leq u_2$, for all $x \in B$

Now $x \in AUB$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow w_1 \leq x \leq u_1$ or $w_2 \leq x \leq u_2$

We choose w = minimum (w_1, w_2)

u = maximum (u_1, u_2)

Thus $x \in AUB$

$\Rightarrow w \leq x \leq u$

$\Rightarrow$ AUB is bounded.

(ii) Try yourself.

Theorem 1.6 (i) The supremum of a set, if it exists, is unique

(ii) The infimum of a set, if it exists is unique.

Proof : (i) Let $A \subseteq \underline{\underline{R}}$

If possible, let there be two suprema, say, u_1 and u_2 of A.

$\because$ u_1 is supremum, by definition,

$u_1 \leq u_2$ ... (1)

Again, $\because$ u_2 is supremum, by definition, $u_2 \leq u_1$ ... (2)

From (1) and (2), $u_1 = u_2$

Thus the supremum is unique

(ii) Try yourself

Note : 1. If a set is bounded above, there exist infinitely many upper bounds. But the least upper bound (supremum) is unique.

2. If a set is bounded below, there exist infinitely many lower bounds. But the greatest lower bound (infimum) is unique.

Do you Know?

According to The Chambers Dictionary an axiom is a self-evident truth, a universally received principle an assumption made for the purpose of argument.

Completeness axiom (or the supremum axiom)

Every nonempty subset of $\underline{\underline{R}}$ that is bounded above has the least upper bound (supremum) in $\underline{\underline{R}}$.

Note : Every nonmepty subset of $\underline{\underline{R}}$ that is bounded below has the greatest lower bound (infimum) in $\underline{\underline{R}}$. It can be proved using the completeness axiom.

Theorem 1.7: A nonempty subset of $\underline{\underline{R}}$ that is bounded below has the greatest lower bound in $\underline{\underline{R}}$

Proof Let S be a nonempty subset of $\underline{\underline{R}}$ that is bounded below.

Let w be a lower bound of S.

$\therefore$ $w \leq x, \forall x \in S$

Let $T = \{-x : x \in S\}$

$\because$ $S \neq \phi$, $\therefore T \neq \phi$.

We have $w \leq x, \forall x \in S$

$\Rightarrow -w \geq -x, \forall x \in S$

$\Rightarrow -x \leq -w, \forall x \in S$

$\therefore$ T is bounded above with – x as an upper bound.

Thus, T is nonempty and it is bounded above.

$\therefore$ by completeness axiom.

T has two least upper bound is $\underline{\underline{R}}$

Let it be M

$\therefore -x \leq M, \forall x \in S$

$\Rightarrow x \geq -M, \forall x \in S$

$\Rightarrow -M \leq x, \forall x \in S$

$\Rightarrow$ – M is a lower bound of S.

Let w' be any lower bound of S.

$\therefore$ – w' is an upper bound of T.

$\therefore$ M = lub (T), $\therefore M \leq -w'$

$\Rightarrow -M \geq w'$

$\Rightarrow w' \leq -M$

$\therefore -M = \text{glb (S)}$.

Thus, S has the greatest lower bound in $\underline{\underline{R}}$

Definition : An ordered field is said to be complete ordered field if the completeness axiom holds in it.

Thus, $\underline{\underline{R}}$ is a complete ordered field.

Note : The property of completeness does not hold good for the ordered field of rational numbers.

Example 7. Let the subset S of $\underline{\underline{Q}}$ be defined by

S = { $x \in \underline{\underline{Q}}$: x > 0 and $x^2 < 2$} Show that S is a nonempty subset of $\underline{\underline{Q}}$ bounded does not belong to $\underline{\underline{Q}}$.

Solution : We have 1 > 0, and $1^2 < 2$

$\therefore 1 \in S$, i.e., $S \neq \phi$.

Clearly 2 is an upper bound S.

$\therefore$ S is bounded above.

If possible, let lub (S) $\in \underline{\underline{Q}}$.

Let lub (S) = u

Then u > 0, and $u \in \underline{\underline{Q}}$.

By the law of trichotomy, there are three possibilitries,

$u^2 < 2$ or $u^2 = 2$ or $u^2 > 2$

Case1. Let $u^2 < 2$

We define a rational number r by

$$r = \frac{4+3u}{3+2u}$$

Clearly r > 0, $\because u > 0$

Again $r - u = \frac{4+3u}{3+2u} - u$

$$= \frac{4+3u-u(3+2u)}{3+2u} = \frac{2(2-u^2)}{3+2u}$$

$\therefore u^2 < 2$, $\therefore 2 > u^2$, i.e $2 - u^2 > 0$

$$\therefore r - u = \frac{2(2-u^2)}{3+2u} > 0$$

i.e., r > u

Again $r^2 - 2 = \left(\frac{4+3u}{3+2u}\right)^2 - 2$

$$= \frac{(4+3u)^2 - 2(3+2u)^2}{(3+2u)^2}$$

$$= \frac{u^2 - 2}{(3+2u)^2}$$

$\because u^2 < 2, \therefore u^2 - 2 < 0$

$\therefore r^2 - 2 < 0$, i.e., $r^2 < 2$

We see that $r \in S$, and r is greater than u, i.e., the lub (S), which is a contradiction.

$\therefore u^2 < 2$ is not possible

Case 2. If possible, let $u^2 = 2$.

This shows that there exists a rational number whose square is 2, which is a contradiction.

$\therefore u^2$ cannot be 2

Case 3. Let $u^2 > 2$

We consider the rational number r defined in case 1.

$r > 0$, and $r - u = \frac{2(2-u^2)}{3+2u}$, from case 1

$\therefore u^2 > 2, \therefore 2 < u^2$, i.e., $2 - u^2 < 0$

$\therefore r - u < 0$, i.e., $r < u$

Again $r^2 - 2 = \frac{2-u^2}{(3+2u)^2}$, from case1

$$\therefore 2 - r^2 = = \frac{u^2 - 2}{(3+2u)^2} < 0$$

i.e., $2 - r^2 < 0$, i.e., $2 < r^2$, i.e., $r^2 > 2$

This shows that r is an upper bound which does not belong to S. Again r is less than u, i.e., the lub (S), which is a contradiction.

$\therefore u^2 > 2$ is not possible.

Thus, none of the possibilities holds good.

So, our supposition is wrong.

$\therefore$ u, i.e., lub (S) does not belong to $\underline{\underline{Q}}$.

Note : $\underline{\underline{R}}$ is a complete ordered field but $\underline{\underline{Q}}$ is not order-complete.

Do You Know?

1. If S is a nonempty set bounded above, the least upper bound of S may or may not belong to S.
2. If S is a nonempty set bounded below, the greatest lower bound of S may or may not belong to S.
3. If S is a nonempty finite set, the least upper bound and the greatest lower bound of S both exist and belong to S.

Theorem 1.8

If S is a nonempty bounded subset of $\underline{\underline{R}}$,

Then glb (S) $\leq$ lub (S)

Proof : $\because$ S is a nonempty bounded subset of $\underline{\underline{R}}$, it has least upper bound and greatest lower bound in $\underline{\underline{R}}$ (by completeness axiom)

Let a $\in$ S

$\therefore$ by definition glb (S) $\leq$ a ... (1)

and a $\leq$ lub (S) ... (2)

From (1) and (2)

glb (S) $\leq$ lub (S)

Theorem 1.9:

If A, and B are two nonempty subsets of $\underline{\underline{R}}$ with the property that $a \leq b \forall a \in A$, b $\in$ B, then

lub (A) $\leq$ glb (B)

Proof : Let b $\in$ B be arbitrary

Now $a \leq b, \forall a \in A$

$\therefore$ b is an upper bound of A, and so, by completeness axiom, it has the lub in $\underline{\underline{R}}$

Again let a $\in$A be arbitrary

Now $a \leq b, \forall b \in B$

$\therefore$ a is a lower bound of B. and so, it has the glb in $\underline{\underline{R}}$

Again a $\leq$ glb (B), $\forall a \in A$

$\therefore$ glb (B) is an upper of A

$\therefore$ lub (A) $\leq$ glb (B)

Theorem 1.10:

Let A, B be nonempty subsets of $\underline{\underline{R}}$ such that A is a subset of B, and

B is bounded above. Then A is bounded above, and

lub (A) $\leq$ lub (B)

Proof : $\because$ B is bounded above, by completeness axiom, the least upper bound of B exists in $\underline{\underline{R}}$

Let $x \in A$

$\therefore A \subseteq B$, $x \in A \Rightarrow x \in B$

$\therefore x \leq$ lub (B).

$\therefore$ A is bounded above by lub (B)

$\therefore$ by completeness axiom, A has the least upper bound in $\underline{\underline{R}}$

$\therefore$ lub (A) $\leq$ lub (B)

Do You Know?

The greatest lower bound or the least upper bound of a set may or may not belong to the set. If the least upper bound of a set belong to the set, it is called the greatest member of the set. Similarly, if the greatest lower bound of a set belongs to the set, it is called the least member of the set.

Theorem 1.11: (A) Let S be a nonempty subset of $\underline{\underline{R}}$ bounded above. Then a real number u is the least upper bound of S if and only if for any $\varepsilon > 0$, there exists $x \in S$ such that

$$u - \varepsilon < x \leq u$$

(B) Let S be a nonempty subset of $\underline{\underline{R}}$ bounded below. Then a real number w is the greatest lower bound of S if and only if for only $\varepsilon > 0$, there exists $x \in S$ such that

$$w \leq x < w + \varepsilon$$

Proof: (A) Let u = lub (S)

$\therefore x \leq u, \forall x \in S$ ... (1)

Let $\varepsilon > 0$ then $u\ \varepsilon < u$ and, therefore, $u - \varepsilon$ cannot be an upper bound of S.

$\therefore$ There exists some $x \in S$ such that $x > \varepsilon - u$ ... (2)

Conversely, let u be any real number satisfying the condirion.

$$u - \varepsilon < x \leq u$$

We are to prove that u = lub (S)

Now $x \leq u$, $\forall x \in S$, (from the given condition)

$\therefore$ u is an upper bound of S.

Let u' be any real number less than u.

$\therefore u - u' > 0$

Let $\varepsilon = u - u' > 0$

$\therefore$ There exists some $x \in S$ such that $x > u - \varepsilon$, (from the given condition)

i.e., $x > u - (u - u')$

i.e., $x > u'$, for same $x \in S$

$\therefore$ u' cannot be an upper bound of S

Thus, we see that no real number, less than u, cannot be an upper bound of S.

$\therefore$ u is the least upper bound of S.

(B) Try yourself.



CHECK YOUR PROGRESS 2

Q.1. Prove that the set of natural numbers $\underline{\underline{N}}$ is not bounded above.

Q. 2. Let A and B be two nonempty subsets of $\underline{\underline{R}}$, and let

C = {x + y : x $\in$ A and y $\in$ B}

If each of A and B has a supermum prove that C has a supremum and sup C = sup A + sup B

Q.3. Let A and B be two nonempty sets such that A, B are bounded above. Prove that AUB is bounded above, and that

sup (AUB) = max (sup (A), sup (B))

Q. 4. Let A and B be two nonempty sets such that A, B are bounded below, and $A \cap B$ is nonempty. Prove that $A \cap B$ is bounded below and max (inf (A), inf (B)) $\leq$ inf $(A \cap B)$

Q.5. Let A be a nonempty subset of $\underline{\underline{R}}$ such that A is bounded below. Let $c \in \underline{\underline{R}}$.

Define the set

$c \in \underline{\underline{R}}$ = $\{c + x : x \in A\}$

Prove that c +A is bounded below and

$\text{glb}(c + A) = c + \text{glb}(A)$

1.7 ARCHIMEDEAN PROPERTY OF REAL NUMBERS

If x, y are two positive real numbers, then there exists a natural number n such that ny > x

Proof : If possible, let there exist no natural number n for which ny > x

Then for every natural number m, my $\leq$ x.

Let us consider S defined by S = {my : $y \in \underline{\underline{N}}$ }

Clearly S is nonempty, and bounded above. x is an upper bound.

$\therefore$ by completeness axiom. S has the least upper bound. Let if be u, i.e., lub (S) = u

$\therefore$ my $\leq$ u, for all $m \in \underline{\underline{N}}$

$\because$ y > 0, u – y < u, and, therefore, u – y cannot be an upper bound of S.

$\therefore$ there exists a natural number p such that

$u - y < py \leq u$

This $\Rightarrow u < y + py$

$\Rightarrow (p+1)y > u$ ..... (1)

Now $p \in \underline{\underline{N}} \Rightarrow p+1 \in \underline{\underline{N}} \Rightarrow (p+1)y \in S$

From (1), we see that u is not the least upper bound of S, which is a contradiction. So, our supposition is wrong.

$\therefore$ ny > x

Note : The above property was named after the ancient Greek mathematician Archimedes by the Austrian mathematician otto stolz (1842-1905). The Archimedean property appears in Elements of Euclid (Definition 4, Book V). It states that, 'Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.'

$\underline{\underline{R}}$ and $\underline{\underline{Q}}$ are Archimedean ordered field. $\underline{\underline{R}}$ is complete, but $\underline{\underline{Q}}$ is not complete.

Corollaries

1. If x is a real number, there exists a natural number n such that n > x

Proof : Let x > 0

By Archimedean property, we have ny > x, where x, y are positive real numbers, and n is a natural number. Taking y = 1, we get n > x.

Let $x \leq 0$

We take n = 1

2. If x is a positive real number, there exists a natural number n such that $0 < \frac{1}{n} < x$

Proof : By Archimedean property, we know that if x, y are positive real numbers, there exists a natural number n such that

$nx > y$

We take y = 1 $\therefore nx > 1$, i.e., $x > \frac{1}{n}$, i.e., $\frac{1}{n} < x$

Again $n > 0 \Rightarrow \frac{1}{n} > 0$, i.e., $0 < \frac{1}{n}$.

Thus, $0 < \frac{1}{n} < x$

Do You Know?

The well-ordering principle states that every nonempty set of positive integers contains a least element.

3. If x is apositive real number, there exists a natural number k such that $k - 1 \leq x < k$

Proof : By Archimedeam property, we know that if x, y are positive real numbers, there exists a natural number n such that ny > x

We take y = 1 $\therefore$ n > x

Let us consider the set

$E = \{m \in \underline{\underline{N}} : m > x\}$

Since n > x, $\therefore n \in E$

$\therefore$ E is nonempty.

$\therefore$ by well-ordering principle E contains a least element.

Let it be k.

$\because k \in E$, $\therefore$ k > x

Again k is the least element in E.

$\therefore$ k – 1 $\geq$ x

$\therefore$ k – 1 $\leq$ x < k

4. If x is a real number, there exists an integer k such that k – 1 $\leq$ x < k

Proof : (i) Let x > 0

Then the result is trun from Cor 3

(ii) Let x = 0

We take k = 1

(iii) Let $x < 0$

We assume that x is not a negative integer

Now $-x > 0$

$\therefore$ from (i), we get that there exists a natural number in such that

$m - 1 < -x < m$

$-x < m$

$\Rightarrow x > -m$

$\Rightarrow -m < x$ and $m - 1 < -x \Rightarrow x < -m + 1$

$\therefore -m < x < -m + 1$

Let $k = -m + 1$

$\therefore$ m is a natural number, k is an integer less than or equal to zero.

$\therefore$ We have $k - 1 < x < k$

If x is a negative inyeger, then $x = k - 1$

Thus $k - 1 \leq n < k$

1.8 DENSITY PROPERTY OF REAL NUMBERS

1.8.1 Density of the Rationals

If x, y are real numbers with $x < y$, there exists a rational number r such that $x < r < y$.

Proof : We have $x < y$, i.e., $y > x$,

$\therefore y - x > 0$

$\therefore$ by Archimedean property of real numbers (Cor 2), there exists a natural number n such that

$$0 < \frac{1}{n} < y - x$$

or, 0 < <ny - nx

or, $ny - nx > 1$

or, $nx + 1 < ny$ ... (1)

Now $nx \in \underline{\underline{R}}$

$\therefore$ by Archimedean property of real numbers (Cor 4), there exists an integer m such that

$m - 1 \leq nx < m$ ... (2)

Now $m - 1 \leq nx \Rightarrow m \leq nx + 1$ ... (2)

$\therefore$ from (1) and (2)

$m \leq nx + 1 < ny$ ..... (3)

Again $nx < m$

$\therefore$ $nx < m$, ny from ..... (3)

or, $x < \frac{m}{n} < y$

Here m is an integer, and n is a natural number.

$\therefore$ $\frac{m}{n}$ is a rational number

Taking $\frac{m}{n} = r$

We get $x < r < y$.

1.8.2 Density of the Irrationals

If x, y are real numbers with $x < y$, there exsits and irrational number t such that $x < t < y$.

Proof : We have

$x < y$

$\therefore \sqrt{2}x < \sqrt{2}y$

Now $\sqrt{2}x$, $\sqrt{2}y$ are real numbers.

$\therefore$ by 1. 8.1, there exists a rational number r such that

$\sqrt{2}x < r < \sqrt{2}y$

or $x < \frac{r}{\sqrt{2}} < y$

Clearly $\frac{r}{\sqrt{2}}$ is irratioral

Taking $\frac{r}{\sqrt{2}} = t$, we get

$x < t < y$

1.9 EXTENDED SET OF REAL NUMERS

We know the two symbols, viz, $+\infty$ or ∞ (plus infinity) and $-\infty$ (minius infinity). These are not real numbers. So, any result stated for real

numbers cannot be applied to ∞ and $-\infty$.

The set of real numbers $\underline{\underline{R}}$ can be extended by adjoining the two symbols by $\underline{\underline{R}}_{\infty}$, Thus

$$\underline{\underline{R}}_{\infty} = \underline{\underline{R}} \cup \{-\alpha, +\alpha\}$$

In $\underline{\underline{R}}_{\infty}$, we define

(i) for all real x, $x + \infty = \infty + x = \infty$

$x(-\infty) = (-\infty) + x = -\infty$

(ii) for $x > 0$, $x.\infty = \infty.x = \infty$

$x.(-\infty) = (-\infty).x = \infty$

(iii) for $x < 0$, $x.\infty = \infty x = -\infty$

$x(-\infty) = (-\infty).x = \infty$

(iv) $\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty$

$\infty.\infty = \infty$

$(-\infty).\infty = \infty.(-\infty) = -\infty$

The order relation in $\underline{\underline{R}}_{\infty}$ satisfies the inequality $-\infty < x < \infty, \forall x \in \underline{\underline{R}}$

If $x > 0$, $0 < x < \infty$

If $x < 0$, $-\infty < x < 0$

Let a, b, be two real numbers. Then

$[a, \infty[= \{x \in \underline{\underline{R}} : x \geq a\}$

$]a, \infty[= \{x \in \underline{\underline{R}} : x \geq a\}$

$]-\infty, b] = \{x \in \underline{\underline{R}} : x \leq b\}$

$]-\infty, b[= \{x \in \underline{\underline{R}} : x < b\}$

$]-\infty, \infty[= \{x \in \underline{\underline{R}} : -\infty < x < \infty\}$

Note : $\infty + (-\infty)$, $(-\infty) + \infty$, $0.\infty$, $\infty.0$

$0.(-\infty), (-\infty).0, \frac{\infty}{\infty}$ are not defined



1.10 LET US SUM UP

- A nonempty set equipped with one or more binary operations defined on it, is called an algebraic structure. A field is an algebraic structure.
- The set R of real numbers is a field under ordinary addition and multiplication.

- The field of real numbers is an ordered field. The field of rational numbers is also an order field. But the field of complex numbers is not ordered.
- Every nonempty subset of $\underline{\underline{R}}$ that is bounded above has the least upper bound (supermom) in $\underline{\underline{R}}$. This is known as the completeness axiom.
- $\underline{\underline{R}}$ is a complete ordered field.
- If x, y are two positive real numbs, then there exists a natural known as n such that ny > x. This is known as tho Archimedean property of real numbers.
- $\underline{\underline{R}}$ is an Archimedean ordered field.
- If x, y are real numbers with x < y, there exists a rational number r such that x < r < y.
- If x, y are real numbers with x < y, there exists an irrational number t such that x < t<y.



1.11 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progresss 1

Ans to Q No 1:

(i) $|x + y + z|$

$= |(x + y) + z|$

$\leq |x + y| + |z|$

$\leq |x| + |y| \mid |z|$

(ii) $|xyz|$

$= |(xy)z|$

$= |xy| \, |z|$

$= |x| \, |y| \, |z|$

Ans to Q No 2: Try yourself

Ans to Q No 3: Try yourself

Ans to Q No 4: Try yourself

Ans to Q No 5: $|x - y = |x + (-y)|$

$\leq |x| + |y|$

$= |x| + |y|$

i.e., $|x - y| \leq |x| + |y|$

Ans to Q No 6: $a < x < b$

iff $a - \frac{a+b}{2} < x - \frac{a+b}{2} < b - \frac{a+b}{2}$

iff $\frac{2a-a-b}{2} < x - \frac{a+b}{2} < \frac{-b-a-b}{2}$

iff $\frac{a-b}{2} < x - \frac{a+b}{2} < \frac{b-a}{2}$

iff $-\frac{b-a}{2} < x - \frac{a+b}{2} < \frac{b-a}{2}$

iff $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$

Do you Know

The meaning of "if and only if" (shortened as iff) is as follows.
'The truth of either one of the connected statements required the truth of the other.' Logical symbol for "iff" is $\Leftrightarrow$.

Note: In the above example it is shown that if $a < x < b$, then $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$,

and conversely if $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$, then $a < x < b$.

Ans to Q No 7: We have

$$\max\{a, b\} = \begin{cases} a, \text{ if } a > b \\ b, \text{ if } a < b \\ \frac{a+b}{2}, \text{ if } a = b \end{cases}$$

$$\min\{a,b\} = \begin{cases} b, \text{ if } a > b \\ a, \text{ if } a < b \\ \frac{a+b}{2}, \text{ if } a = b \end{cases}$$

Therefor

if a > b, max {a, b} + min {a, b} = a + b

if a < b, max {a, b} + min {a, b) = b + a

if a = b, max {a, b} + min {a, b} = a + b

Agin

if a > b, max {a, b} – min {a, b} = a – b = {a – b|

if a < b, max {a, b} – min {a, b) = b – a = |a – b|

if a = b, max {a, b} – min {a, b} = 0 = |a – b|

Thus, for all real numbers a, and b

max {a, b} + min {a, b} = a + b ... (1)

max {a, b} – min {a, b} = |a – b| ... (2)

Adding (1) and (2), we get

2 max {a, b} = a + b + |a – b|

i.e., max {a, b} = $\frac{1}{2}$ [a + b + |a – b|]

Again subtracting (2) from (1), we get

2 min {a, b} = a + b – |a – b|

i.e., min {a, b} = $\frac{1}{2}$ [a + b – |a – b|]

Ans to Q No 8:

$\left|\frac{x+5}{3x-1}\right| = 4$, (given)

$\therefore \frac{x+5}{3x-1} = 4$ or $\therefore \frac{x+5}{3x-1} = -4$

Now $\frac{x+5}{3x-1} = 4$

$\Rightarrow$ x + 5 = 4 (3x – 1)

$\Rightarrow$ x + 5 = 12x – 4

$\Rightarrow$ 12x – x = 5 + 4

$\Rightarrow$ 11x = 9

$\Rightarrow x = \frac{9}{11}$

Again, $\frac{x+5}{3x-1} = -4$

$\Rightarrow x + 5 = -4(3x - 1)$

$\Rightarrow x + 5 = -12x + 4$

$\Rightarrow x + 12x = 4 - 5$

$\Rightarrow 13x = -1$

$\Rightarrow x = -\frac{1}{13}$

Thus, $x = \frac{9}{11}, -\frac{1}{3}$

Ans to Q No 9: We have

$|a + b| = |a| + |b|$

iff $|a + b|^2 = (|a| + |b|)^2$

iff $(a + b)^2 = |a|^2 + |b|^2 + 2|a|\,|b|$

iff $a^2 + b^2 + 2ab = a^2 + b^2 + 2|ab|$

iff $ab = |ab|$

iff $ab \geq 0$

Ans to Q No 10: We have

$|a + b| < |a| + |b|$

iff $|a + b|^2 < (|a| + |b|)^2$

iff $(a + b)^2 < |a|^2 + |b|^2 + 2|a|\,|b|$

iff $a^2 + b^2 + 2ab < a^2 + b^2 + 2|ab|$

iff $ab < |ab|$

iff $ab < 0$

Ans to Q No 11: (i) We have

$|x - 5| < 2$

$\Leftrightarrow \quad 5 - 2 < x < 5 + 2$

$\Leftrightarrow \quad 3 < x < 7$

$\therefore$ the statement is true.

(ii) $|x - 4| < 1$

$\Leftrightarrow \quad 4 - 1 < x < 4 + 1$

$\Leftrightarrow \quad 3 < x < 5$

$\therefore$ The statement is not true.

(iii) $|x - 1| < 2$

$\Leftrightarrow \quad 2 - 1 < x < 2 + 1$

$\Leftrightarrow$ 1 < x < 3

$\therefore$ the statement is true.

Ans to Q No 12: We know that

$$|x| = \begin{cases} x, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -x, \text{if } x < 0 \end{cases}$$

Here $a \neq 0$

Let a > 0

$$\therefore f(a) = \frac{|a|}{a} = \frac{a}{a} = 1$$

$$f(-a) = \frac{|-a|}{-a} = \frac{a}{-a} = -1$$

$\therefore |f(a) - f(-a)| = |1 - (-1)| = |1 + 1| = |2| = 2$

Let a < 0

$$\therefore f(a) = \frac{|a|}{a} = \frac{-a}{a} = -1$$

$$f(-a) = \frac{|-a|}{-a} = \frac{-a}{-a} = 1$$

$\therefore |f(a) - f(-a)| = |-1 - 1| = |-2| = 2$

Ans to Q No 13: We have

$|a + b| \leq |a| + |b|$

$\Rightarrow 1 + |a + b| \leq 1 + |a| + |b|$

$$\Rightarrow \frac{1}{1+|a+b|} \geq \frac{1}{1+|a|+|b|}$$

$$\Rightarrow -\frac{1}{1+|a+b|} \leq -\frac{1}{1+|a|+|b|}$$

$$\Rightarrow 1-\frac{1}{1+|a+b|} \leq 1-\frac{1}{1+|a|+|b|}$$

$$\Rightarrow \frac{1+|a+b|-1}{1+|a+b|} \leq \frac{1+|a|+|b|-1}{1+|a|+|b|}$$

$$\Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} \text{ ..... (1)}$$

again $1 + |a| + |b| \geq t\,|a|$

$$\Rightarrow \quad \frac{1}{1+|a|+|b|} \le \frac{1}{1+|a|}$$

$$\Rightarrow \quad \frac{|a|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} \qquad \text{... (2)}$$

Similarly $\frac{|b|}{1+|a|+|b|} \le \frac{|b|}{1+|b|}$

Again (2) and (3), we get

$$\frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

$$\Rightarrow \quad \frac{|a|+|b|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \qquad \text{... (4)}$$

From (1) and (4), we get

$$\frac{|a+b|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

Ans to Q No 14: We have

$|a - b| \le |a| + |b|$

$$\Rightarrow \quad 1 + |a - b| \le 1 + |a| + |b|$$

$$\Rightarrow \quad \frac{1}{1+|a-b|} \ge \frac{1}{1+|a|+|b|}$$

$$\Rightarrow \quad -\frac{1}{1+|a-b|} \le -\frac{1}{1+|a|+|b|}$$

$$\Rightarrow \quad 1-\frac{1}{1+|a-b|} \le 1-\frac{1}{1+|a|+|b|}$$

$$\Rightarrow \quad \frac{1+|a-b|-1}{1+|a-b|} \le \frac{1+|a|+|b|-1}{1+|a|+|b|}$$

$$\Rightarrow \quad \frac{|a-b|}{1+|a-b|} \le \frac{|a|+|b|}{1+|a|+|b|} \qquad \text{... (1)}$$

Again $1 + |a| + |b| \ge a + |a|$

$$\Rightarrow \quad \frac{1}{1+|a|+|b|} \le \frac{1}{1+|a|}$$

$$\Rightarrow \quad \frac{|a|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} \qquad \text{... (2)}$$

Similarly $\frac{|b|}{1+|a|+|b|} \le \frac{|b|}{1+|b|}$ ... (3)

Again (2) and (3), we get

$$\frac{|a|}{1+|a|+|b|}+\frac{|b|}{1+|a|+|b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

$$\Rightarrow \quad \frac{|a|+|b|}{1+|a|+|b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|} \qquad \text{... (4)}$$

$$\Rightarrow \quad \frac{|a|+|b|}{1+|a-b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

From (1) and (4), We get

$$\Rightarrow \quad \frac{|a-b|}{1+|a-b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

Check Your Progresss 2

Ans to Q No 1: If possible, Let $\underline{\underline{N}}$ be bounded above. Therefore, by completeness axiom, it has the least upper bound. Let it be u.

$\therefore$ u – 1 is not an upper bound for $\underline{\underline{N}}$.

$\therefore$ There exists a $\in \underline{\underline{N}}$ such that u – 1< a.

i.e., u < 1 + a

But a $\in \underline{\underline{N}} \Rightarrow$ 1 + a $\in \underline{\underline{N}}$

$\therefore$ u is not an upper bound for $\underline{\underline{N}}$, which is a contradiction.

$\therefore$ Our supposition is wrong.

$\therefore$ $\underline{\underline{N}}$ is not bounded above.

Ans to Q No 2: Let u = sup (A)

w = sup (B)

$c = \{x+2 : x \in A \text{ and } y \in B\}$

It is required to prove that u + w = sup (c)

Let z $\in \subset$ be arbitray

Then there exist x $\in$ A and y $\in$ B

Such that

z = x + y

Now x $\le$ u, and y $\le$ w

$\therefore$ x + y $\le$ u + w

$\Rightarrow$ z $\le$ u + w

$\therefore$ u + w is an upper bound for c.

Let a be any real number such that a < u + w

$\therefore$ a – w < u i. e., a – q < sup (A)

$\therefore$ a – w is not an upper bound for A,

$\therefore$ There exists some x $\in$ A such that a – w < x

or a – x < w, i.e., a – x < sup (B)

$\therefore$ a – x is not an upper bound for B.

$\therefore$ There exists some y $\in$ B such that a – x < y

i.e., a < x + y, where x + y $\in$ c.

$\therefore$ a is not an upper bound for c.

$\therefore$ u + w is the least upper bound for c.

Thus, sup (c) = u + w

= sup (A) + sup (B)

Ans to Q No 3: Let u = max (sup (A), sup (B)

Clearly sup (A) $\leq$ u,

sup (B) $\leq$ u

Let $x \in A \cup B$

$\Rightarrow$ x $\in$ A or x $\in$ B (or both)

If x $\in$ A, then x $\leq$ sup (A) $\leq$ u

If x $\in$ B, then x $\leq$ sup (B) $\leq$ u

$\therefore$ $A \cup B$ is bound above, and u is an upper bound fo $A \cup B$

$\therefore$ by completeness axiom, $A \cup B$ has the least upper bound, i.e., sup ($A \cup B$) exists.

Thus, sup ($A \cup B$) $\leq$ u

i.e., sup ($A \cup B$) $\leq$ max (sup (A), sup (B)) ... (1)

We know that $A \subseteq A \cup B$

$\therefore$ sup (A) $\leq$ sup ($A \cup B$), [Theorem1, 10]

Also $B \subseteq A \cup B$

$\therefore$ sup (B) $\leq$ sup ($A \cup B$), [Theorem 1.10]

$\therefore$ max (sup(A), sup (B)) $\leq$ sub ($A \cup B$) ... (2)

From (1) and (2), sup ($A \cup B$) = max (sup (A), sup (B))

Ans to Q No 4: A, B are bounded below

$\therefore$ inf (A), inf (B) exist.

Moreover $A \cap B \neq \phi$

Let x $\in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$x \in A \Rightarrow \inf(A) \leq x$

$x \in B \Rightarrow \inf(B) \leq x$

$\therefore$ inf (A), and inf (B) are both lower bounds for $A \cap B$

$\therefore$ $\inf(A) \leq \inf A \cap B$ ... (1)

$\inf(B) \leq \inf A \cap B$ ..... (2)

From (1) and (2)

max (inf (A), inf (B) $\leq \inf(A \cap B)$

Ans to Q No 5: Let $x \in A$

$\therefore$ glb (A) $\leq x, \forall x \in A$

$\therefore$ $c + glb(A) \leq c + x$

Now $c + x \; c \in +A$

$\therefore$ c + glb (A) is a lower bound for c + A

$\therefore$ $c + glb(A) \leq glb(c + A)$ ..... (1)

Now glb (c + A) $\leq z, \forall z \in c + A$

$\Rightarrow glb(c + A) \leq c + x,$ where $z = c + x, x \in A$

$\Rightarrow glb\ (c + A) - c \leq x, \forall x \in A$

or glb (c + A) $\leq$ c + glb (A) ..... (2)

From (1) and (3)

glb (c + A) = c + glb (A)



1.12 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin

2) Methods of real Analysis by Richand R. Goldbery

3) Mathematical Analysis by S. C. Malik and Savita Arora

4) A Course of Mathematical Analysis by shanti Narayan.



1.13 MODEL QUESTIONS

Q 1: If a, b are real numbers and $\varepsilon > 0$, show that $|a - b| < \varepsilon$ if and only if $b - \varepsilon < a < b + \varepsilon$.

Q 2: If a, b are real numbers, show that

(i) $\sqrt{a^2 + b^2} \leq |a| + |b|$

(ii) $\sqrt{|a + b|} \leq \sqrt{|a|} + \sqrt{|b|}$

Q 3: Let $S = \{x \in \underline{\underline{Q}} : x > 0 \text{ and } x^2 < 3\}$

Show that S is nonempty and bounded above, but it does not have the supremum in $\underline{\underline{Q}}$.

Q 4: Give an example of an ordered field which is not complete. Justify your answer.

Q 5: Find the suprema and infima, if they exist, of the following sets

(i) $\left\{\frac{n-1}{n} : n \in \underline{\underline{N}}\right\}$

(ii) $\left\{\frac{n}{n+1} : n \in \underline{\underline{N}}\right\}$

(iii) $\left\{\frac{1}{n} : n \in \underline{\underline{Z}}, n \neq 0\right\}$

Q 6: Show that every finite subset of the set of real numbers is bounded.

Q 7: Show that every subset of a bounded set is bounded.

Q 8: Let A be a nonempty subset or $\underline{\underline{R}}$ such that A is bounded above.

Let $c \in \underline{\underline{R}}$. Define the set

$cA = \{cx : x \in A\}$

Prove that if $c > 0$, then cA is bounded above and sup (cA) = c sup (A)

Q 9: For any set A, define the set $-A = \{y \in \underline{\underline{R}} : \exists\, x \in A$ such that $y = -x\}$

Prove that if A is nonempty and bounded, then sup (–A) = – inf (A)

Q 10: Prove that between any two real numbers there is an irrational number.

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